

excursions in Modern Mathematics




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2 The Mathematics of Power

2.1 An Introduction to Weighted Voting

2.2 The Banzhaf Power Index

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Weighted Voting

In weighted voting the player's weights can be deceiving. Sometimes a player with a few votes can have as much power as a player with many more.

Sometimes two players have almost an equal number of votes, and yet one player has a lot of power and the other one has none

Banzhaf Power Index

To pursue these ideas further we will need a formal definition of what “power” means and how it can be measured. In this section we will introduce **a mathematical method for measuring the power of the players in a weighted voting system** called the **Banzhaf Power Index**. This method was first proposed in 1965 by, of all people, a law professor named John Banzhaf III.

Coalitions

A **coalition** **is any set of players who might join forces and vote the same way**. In principle, we can have a coalition with as few as *one* player and as many as *all* players. **The coalition consisting of all the players is** called the **grand coalition**.

Coalitions

Since coalitions are just sets of players, the most convenient way to describe coalitions mathematically is to use *set notation*. For example, the coalition consisting of players P_1, P_2, and P_3 can be written as the set $\{P_1, P_2, P_3\}$ (order does not matter).

Winning Coalitions

We call coalitions with enough votes to win **winning coalitions** and coalitions that don't **losing coalitions.**

A single-player coalition can be a winning coalition only when that player is a *dictator*.

(We'll focus on winning coalitions with a minimum of two players.)

At the other end of the spectrum, **the grand coalition is always a winning coalition**, since it controls all the votes. In some weighted voting systems the grand coalition is the only winning coalition.

Critical Players

In a winning coalition, a critical player is **a player where they must have that player's votes to win**.

In other words, if we subtract a critical player's weight from the total weight of the coalition, the number of remaining votes drops below the quota. Sometimes a winning coalition has no critical players, sometimes a winning coalition has several critical players, and when the coalition has just enough votes to make the quota, then every player is critical.

Critical Player

CRITICAL PLAYER

A player P in a winning coalition is a critical player for the coalition if and only if $W - w < q$ (where W denotes the weight of the coalition and w denotes the weight of P).

Notation

The standard notation used to describe a weighted voting system is to use square brackets and inside the square brackets to write the quota q first (followed by a colon) and then the respective weights of the individual players separated by commas. It is convenient and customary to list the weights in numerical order, starting with the highest, and we will adhere to this convention throughout the chapter.

Notation

Thus, a generic weighted voting system with N players can be written as:

GENERIC WEIGHTED VOTING SYSTEM WITH N PLAYERS

$$[q: w_1, w_2, \ldots, w_N]$$

(with $w_1 \geq w_2 \geq \ldots \geq w_N$)

Example 2.8 The Weirdness of Parliamentary Politics

The Parliament of Icelandia has 200 members, divided among three political parties: P_1, P_2, and P_3, with 99, 98, and 3 seats in Parliament, respectively. Decisions are made by majority vote, which in this case requires 101 out of the total 200 votes. Let's assume, furthermore, that in Icelandia members of Parliament always vote along party lines (not voting with your party is very unusual in parliamentary governments).

Example 2.8 The Weirdness of Parliamentary Politics

We can think of the Parliament of Icelandia as the weighted voting system [101: 99, 98, 3].

In this weighted voting system we have four winning coalitions.

TABLE 2-1 Winning Coalitions and Critical Players for [101: 99, 98, 3]

Coalition	Weight	Critical players
$\{P_1, P_2\}$	197	P_1 and P_2
$\{P_1, P_3\}$	102	P_1 and P_3
$\{P_2, P_3\}$	101	P_2 and P_3
$\{P_1, P_2, P_3\}$	200	None

Example 2.8 The Weirdness of Parliamentary Politics

In the two-party coalitions, both parties are critical players (without both players the coalition wouldn’t win); in the grand coalition no party is critical–any two parties together have enough votes to win. Each of the three parties is critical the same number of times, and consequently, one could rightfully argue that each of the three parties has the same amount of power (never mind the fact that one party has only 3 votes!).

Measure of Power

A player’s power should be measured by how often the player is a critical player.

Thus, count the number of winning coalitions in which that player is critical. From Table 2-1 we can clearly see that in [101: 99, 98, 3] each player is critical twice. Since there are three players, each critical twice, we can say that each player holds two out of six, or one-third of the power. The preceding ideas lead us to the final definitions of this section.

The Banzhaf Power Index

We start by counting how many times P_1 is a critical player in a winning coalition. Let's call this number the *critical count* for P_1, and denote it by B_1. We repeat the process for each of the other players and find their respective *critical counts* B_2, B_3,…, B_N . We then let T denote the sum of the critical counts of all the players ($T = B_1 + B_2 + \ldots + B_N$) and compute the ratio B_1/T (critical count for P_1 over the total of all critical counts).

The Banzhaf Power Index

The ratio B_1/T is the **Banzhaf power index** of P_1. This ratio measures the size of P_1's "share" of the "power pie," and can be expressed either as a fraction or decimal between 0 and 1 or equivalently, as a percent between 0 and 100%. For convenience, we will use the symbol β_1 (read "beta-one") to denote the Banzhaf power index of P_1. (Repeat the process for each of the other players.)

The Banzhaf Power Distribution

Just like P_1, each of the other players in the weighted voting system has a Banzhaf power index, which we can find in a similar way. The complete list of power indexes $\beta_1, \beta_2, \ldots, \beta_N$, is called the **Banzhaf power distribution** of the weighted voting system.

The sum of all the β's is 1 (or 100% if they are written as percentages).

Calculate the Banzhaf Power Distribution

COMPUTING THE BANZHAF POWER DISTRIBUTION OF A WEIGHTED VOTING SYSTEM

Step 1. Make a list of all possible *winning* coalitions.

Step 2. Within each winning coalition determine which are the critical players. (For record-keeping purposes, it is a good idea to underline each critical player.)

Calculate the Banzhaf Power Distribution

Step 3. Count the number of times that P_1 is critical. This gives B_1, the critical count for P_1. Repeat for each of the other players to find $B_2, B_3, \ldots, B_N$.

Step 4. Add all the *B*'s in Step 3. Let's call this number *T*. ($T = B_1 + B_2 + \ldots + B_N$ represents the total of all critical counts.)

Calculate the Banzhaf Power Distribution

Step 5. Find the ratio $\beta_1 = B_1/T$.

This gives the *Banzhaf power index* of P_1. Repeat for each of the other players to find β_2, β_3,…,β_N. The complete list of β's is the *Banzhaf power distribution* of the weighted voting system.

Strategy: Finding Winning Coalitions

The most demanding step is Step 1. When we have only three players, we can list the winning coalitions on the fly–there simply aren’t that many–but as the number of players increases, the number of possible winning coalitions grows rapidly. This is important because if we miss a single one, we are in all likelihood going to get the wrong Banzhaf power distribution. One conservative strategy is to make a list of all possible coalitions and then cross out the losing ones.

Ex 2.10 Banzhaf Power and the NBA Draft

In the Flyers draft system the head coach (P_1) has 4 votes,the general manager (P_2) has 3 votes, the director of scouting operations (P_3) has 2 votes, and the team psychiatrist (P_4) has 1 vote. Of the 10 votes cast, a simple majority of 6 votes is required for a yes vote on a player to be drafted. In essence, the Flyers operate as the weighted voting system **[6: 4, 3, 2, 1].** We will now find the Banzhaf power distribution of this weighted voting system using Steps 1 through 5.

Ex 2.10 Banzhaf Power and the NBA Draft

Step 1. Table 2-2 starts with the complete list of all possible coalitions and their weights.

TABLE 2-2 Coalitions for [6: 4, 3, 2, 1]

Coalition (winning = *)	Weight	Coalition (winning = *)	Weight
$\{\underline{P_1}, \underline{P_2}\}$*		$\{P_1, P_2, P_3\}$*	
$\{\underline{P_1}, \underline{P_3}\}$*		$\{\underline{P_1}, \underline{P_2}, P_4\}$*	
$\{P_1, P_4\}$		$\{\underline{P_1}, \underline{P_3}, P_4\}$*	
$\{P_2, P_3\}$		$\{\underline{P_2}, \underline{P_3}, \underline{P_4}\}$*	
$\{P_2, P_4\}$		$\{P_1, P_2, P_3, P_4\}$*	
$\{P_3, P_4\}$			

Ex 2.10 Banzhaf Power and the NBA Draft

Step 2. Disregard losing coalitions. Determine which players are critical. Underline them.

TABLE 2-2 Coalitions for [6: 4, 3, 2, 1]

Coalition (winning = *)	Weight	Coalition (winning = *)	Weight
$\{\underline{P_1}, \underline{P_2}\}^*$	7	$\{P_1, P_2, P_3\}^*$	9
$\{\underline{P_1}, \underline{P_3}\}^*$	6	$\{\underline{P_1}, \underline{P_2}, P_4\}^*$	8
$\{P_1, P_4\}$	5	$\{\underline{P_1}, \underline{P_3}, P_4\}^*$	7
$\{P_2, P_3\}$	5	$\{\underline{P_2}, \underline{P_3}, \underline{P_4}\}^*$	6
$\{P_2, P_4\}$	4	$\{P_1, P_2, P_3, P_4\}^*$	10
$\{P_3, P_4\}$	3		

Ex 2.10 Banzhaf Power and the NBA Draft

Step 3. Tally how many times each player is underlined. Critical counts: $B_1 = 5$, $B_2 = 3$, $B_3 = 3$, and $B_4 = 1$.

TABLE 2-2 Coalitions for [6: 4, 3, 2, 1]

Coalition (winning = *)	Weight	Coalition (winning = *)	Weight
$\{\underline{P_1}, \underline{P_2}\}$*	7	$\{P_1, P_2, P_3\}$*	9
$\{\underline{P_1}, \underline{P_3}\}$*	6	$\{\underline{P_1}, \underline{P_2}, P_4\}$*	8
$\{P_1, P_4\}$	5	$\{\underline{P_1}, \underline{P_3}, P_4\}$*	7
$\{P_2, P_3\}$	5	$\{\underline{P_2}, \underline{P_3}, \underline{P_4}\}$*	6
$\{P_2, P_4\}$	4	$\{P_1, P_2, P_3, P_4\}$*	10
$\{P_3, P_4\}$	3		

Ex 2.10 Banzhaf Power and the NBA Draft

Step 4. $T = 5 + 3 + 3 + 1 = 12$

Step 5. $\beta_1 = \frac{5}{12} = 41\frac{2}{3}\%$ $\beta_2 = \frac{3}{12} = 25\%$

$\beta_3 = \frac{3}{12} = 25\%$ $\beta_4 = \frac{1}{12} = 8\frac{1}{3}\%$

Interesting and unexpected: the team's general manager (P_2) and the director of scouting operations (P_3) have the same Banzhaf power index–not exactly the arrangement originally intended.

A Brief Mathematical Detour

For a given number of players, how many different coalitions are possible? Here, our identification of coalitions with sets will come in particularly handy. Except for the empty subset { }, we know that every other subset of the set of players can be identified with a different coalition. This means that we can count the total number of coalitions by counting the number of subsets and subtracting one. So, how many subsets does a set have?

A Brief Mathematical Detour

Each time we add a new element we are doubling the number of subsets.

TABLE 2-3 The Subsets of a Set

Set	$\{P_1, P_2\}$	$\{P_1, P_2, P_3\}$		$\{P_1, P_2, P_3, P_4\}$		$\{P_1, P_2, P_3, P_4, P_5\}$
Number of subsets	4	8		16		32
Subsets	$\{\ \}$	$\{\ \}$	$\{P_3\}$	$\{\ \}$	$\{P_4\}$	The 16 subsets of $\{P_1, P_2, P_3, P_4\}$ along with each of these with P_5 thrown in.
	$\{P_1\}$	$\{P_1\}$	$\{P_1, P_3\}$	$\{P_1\}$	$\{P_1, P_4\}$	
	$\{P_2\}$	$\{P_2\}$	$\{P_2, P_3\}$	$\{P_2\}$	$\{P_2, P_4\}$	
	$\{P_1, P_2\}$	$\{P_1, P_2\}$	$\{P_1, P_2, P_3\}$	$\{P_1, P_2\}$	$\{P_1, P_2, P_4\}$	
				$\{P_3\}$	$\{P_3, P_4\}$	
				$\{P_1, P_3\}$	$\{P_1, P_3, P_4\}$	
				$\{P_2, P_3\}$	$\{P_2, P_3, P_4\}$	
				$\{P_1, P_2, P_3\}$	$\{P_1, P_2, P_3, P_4\}$	

A Brief Mathematical Detour

Since each time we add a new player we are doubling the number of subsets, we will find it convenient to think in terms of powers of 2.

TABLE 2-4 The Number of Coalitions

Players	Number of subsets	Number of coalitions
P_1, P_2	$4 = 2^2$	$2^2 - 1 = 3$
P_1, P_2, P_3	$8 = 2^3$	$2^3 - 1 = 7$
P_1, P_2, P_3, P_4	$16 = 2^4$	$2^4 - 1 = 15$
P_1, P_2, P_3, P_4, P_5	$32 = 2^5$	$2^5 - 1 = 31$
⋮	⋮	⋮
$P_1, P_2, \ldots, P_N$	2^N	$2^N - 1$

Shortcuts for Computing Banzhaf Power Distributions

We will now return to the problem of computing Banzhaf power distributions. Given what we now know about the rapid growth of the number of coalitions, the strategy used in Example 2.10 (list all possible coalitions and then eliminate the losing ones) can become a little tedious (to say the least) when we have more than a handful of players. Sometimes we can save ourselves a lot of work by figuring out directly which are the winning coalitions.

Example 2.11 Winning Coalitions Rule

The disciplinary committee at George Washington High School has five members: the principal (P_1), the vice principal (P_2), and three teachers (P_3, P_4, and P_5). When voting on a specific disciplinary action the principal has three votes, the vice principal has two votes, and each of the teachers has one vote. A total of five votes is needed for any disciplinary action. Formally speaking, the disciplinary committee is the weighted voting system [5: 3, 2, 1, 1, 1].

Example 2.11 Winning Coalitions Rule

There are $2^5 - 1 = 31$ possible coalitions. Subtract the five one-player coalitions, we are left with 26 coalitions. Skip losing coalitions and list only the winning coalitions. Organize: we will go through the winning coalitions systematically according to the number of players in the coalition.There is only one two-player winning coalition, namely $\{P_1, P_2\}$. The only three-player winning coalitions are those that include the principal P_1. All four-player coalitions are winning coalitions, and so is the grand coalition.

Example 2.11 Winning Coalitions Rule

Steps 1 and 2.

The winning coalitions, with the critical players underlined.

TABLE 2-5 Winning Coalitions in [5: 3, 2, 1, 1, 1] and Critical Players

Winning coalitions	Comments
$\{\underline{P_1}, \underline{P_2}\}$	Only possible winning two-player coalition.
$\{\underline{P_1}, \underline{P_2}, P_3\}$ $\{\underline{P_1}, \underline{P_2}, P_4\}$ $\{\underline{P_1}, \underline{P_2}, P_5\}$ $\{\underline{P_1}, \underline{P_3}, \underline{P_4}\}$ $\{\underline{P_1}, \underline{P_3}, \underline{P_5}\}$ $\{\underline{P_1}, \underline{P_4}, \underline{P_5}\}$	Winning three-player coalitions must contain P_1 plus any two other players.
$\{\underline{P_1}, P_2, P_3, P_4\}$ $\{\underline{P_1}, P_2, P_3, P_5\}$ $\{\underline{P_1}, P_2, P_4, P_5\}$ $\{\underline{P_1}, P_3, P_4, P_5\}$ $\{\underline{P_2}, \underline{P_3}, \underline{P_4}, \underline{P_5}\}$	All four-player coalitions are winning coalitions.
$\{P_1, P_2, P_3, P_4, P_5\}$	The grand coalition always wins.

Example 2.11 Winning Coalitions Rule

Step 3. The critical counts are $B_1 = 11$, $B_2 = 5$, $B_3 = 3$, $B_4 = 3$, and $B_5 = 3$.

Step 4. $T = 25$

Step 5. $\beta_1 = \frac{11}{25} = 44\%$ $\beta_2 = \frac{5}{25} = 20\%$

$$\beta_3 = \beta_4 = \beta_5 = \frac{3}{25} = 12\%$$