

excursions in Modern Mathematics




Peter Tannenbaum

seventh edition

3 The Mathematics of Sharing

The Lone-Chooser Method

A completely different approach for extending the divider-chooser method was proposed in 1964 by A.M.Fink, a mathematician at Iowa State University. In this method one player plays the role of chooser, all the other players start out playing the role of dividers. For this reason, the method is known as the lone-chooser method. Once again, we will start with a description of the method for the case of three players.

The Lone-Chooser Method for 3 Players

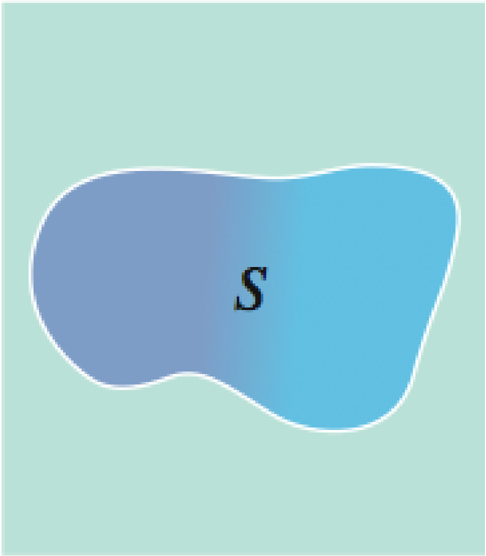
Preliminaries

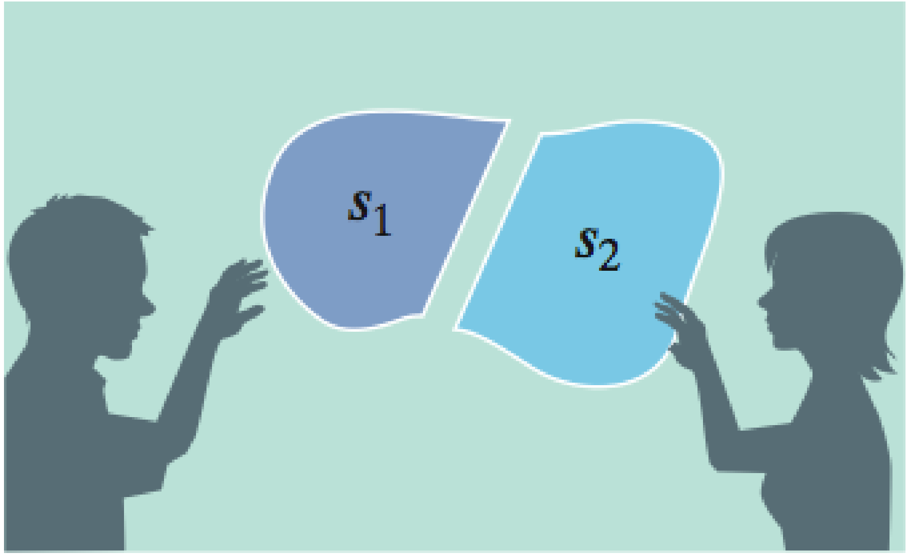
We have one chooser and two dividers. Let's call the chooser C and the dividers D_1 and D_2. As usual, we decide who is what by a random draw.

The Lone-Chooser Method for 3 Players

Step 1 (Division)

D_1 and D_2 divide S between themselves into two fair shares. To do this, they use the divider-chooser method. Let's say that D_1 ends up s_1 with D_2 and ends up with s_2.

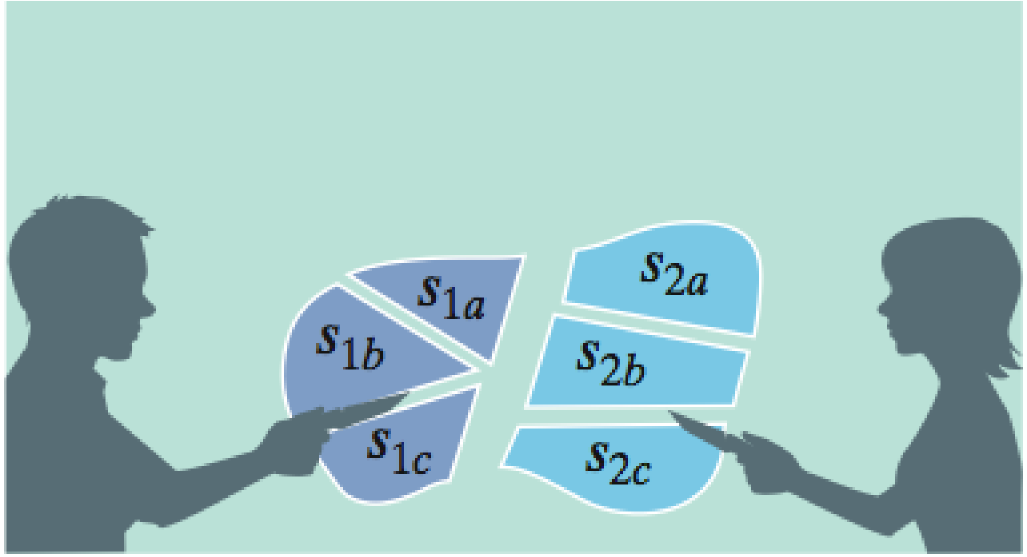




The Lone-Chooser Method for 3 Players

Step 2 (Subdivision)

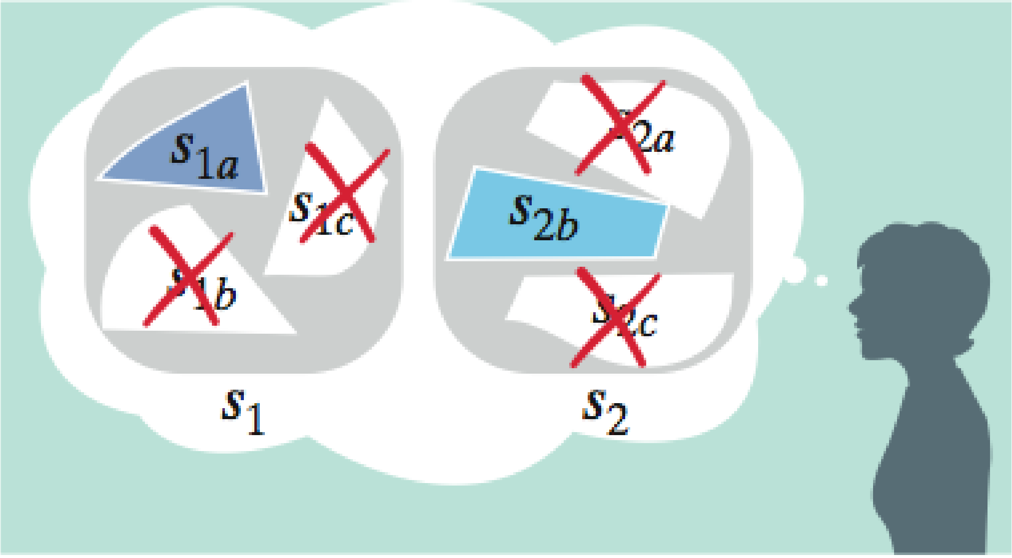
Each divider divides his share into three subshares. Thus, D_1 divides s_1 into three subshares, which we will call s_{1a}, s_{1b}, and s_{1c}. Likewise, D_2 divides s_2 into three subshares, which we will call s_{2a}, s_{2b}, and s_{2c}.



The Lone-Chooser Method for 3 Players

Step 3 (Selection)

The chooser C now selects one of D_1's three subshares and one of D_2's three subshares (whichever she likes best). These two subshares make up C's final share. D_1 then keeps the remaining two subshares from s_1, and D_2 keeps the remaining two subshares from s_2.



The Lone-Chooser Method for 3 Players

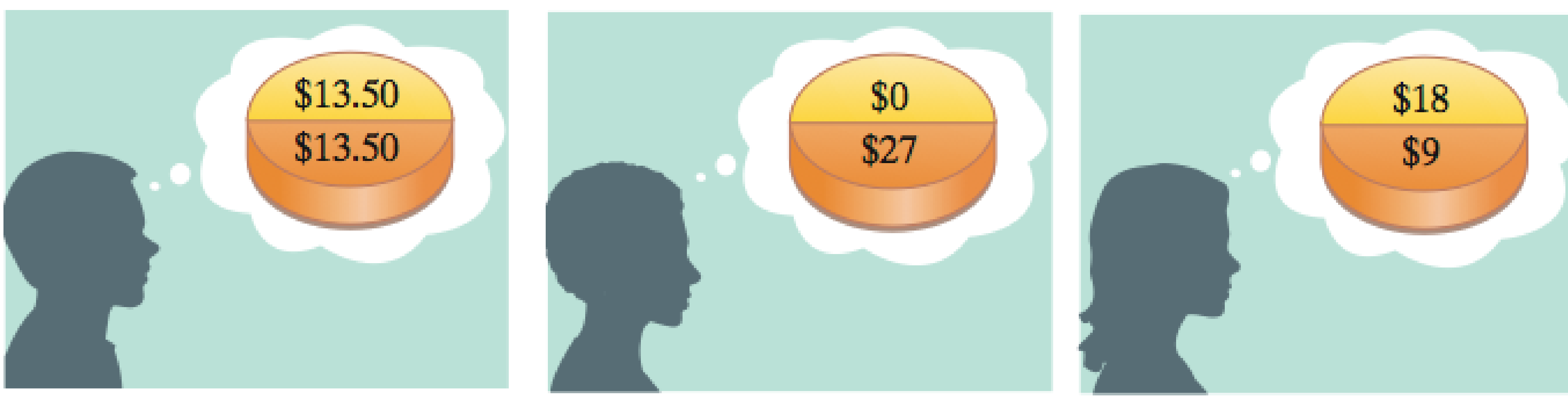
Why is this a fair division of *S*? D_1 ends up with two-thirds of s_1. To D_1, s_1 is worth at least one-half of the total value of *S*, so two-thirds of s_1 is at least one- third–a fair share. The same argument applies to D_2. What about the chooser's share? We don't know what s_1 and s_2 are each worth to *C*, but it really doesn't matter–a one-third or better share of s_1 plus a one-third or better share of s_2 equals a one-third or better share of $(s_1 + s_2)$ and thus a fair share of the cake.

Example 3.7 Lone Chooser with 3 Players

David, Dinah, and Cher are dividing an orange-pineapple cake using the lone-chooser method. The cake is valued by each of them at $27, so each of them expects to end up with a share worth at least $9. Their individual value systems (not known to one another, but available to us as outside observers) are as follows:

Example 3.7 Lone Chooser with 3 Players

- David likes pineapple and orange the same.
- Dinah likes orange but hates pineapple.
- Cher likes pineapple twice as much as she likes orange.

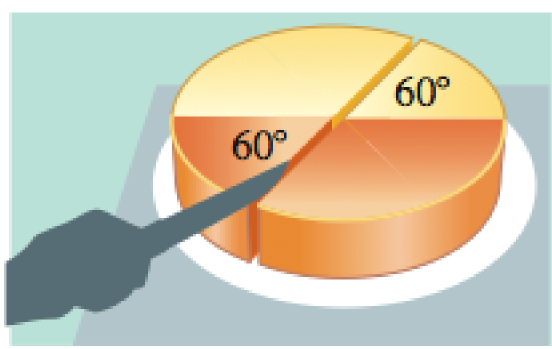


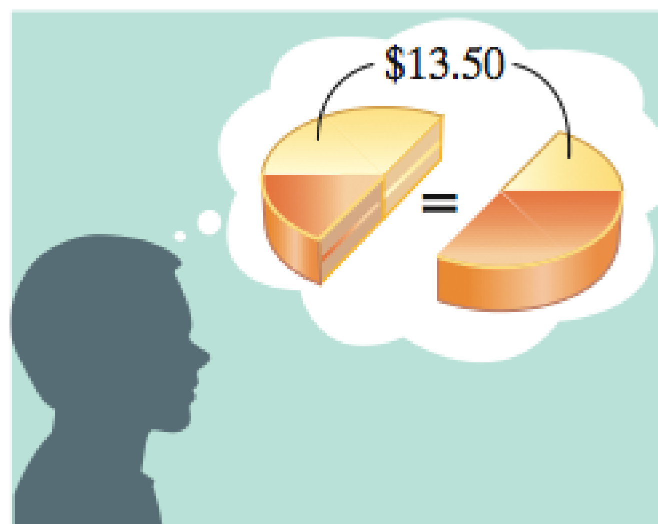
Example 3.7 Lone Chooser with 3 Players

After a random selection, Cher gets to be the chooser and thus gets to sit out Steps 1 & 2.

Step 1 (Division)

David and Dinah start by dividing the cake between themselves using the divider-chooser method. After a coin flip, David cuts the cake into two pieces.





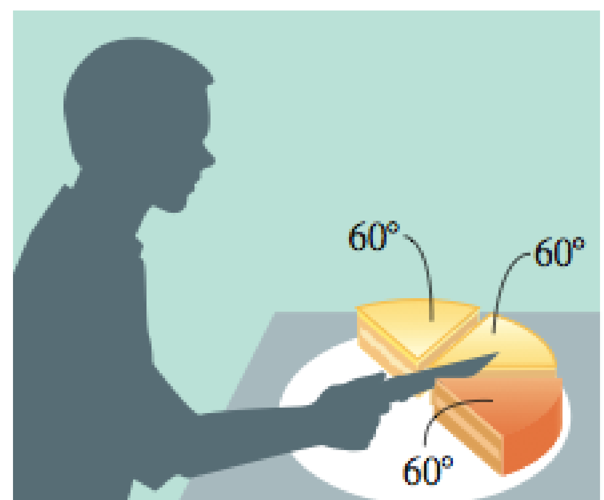
Example 3.7 Lone Chooser with 3 Players

Step 1 (Division) continued

Since Dinah doesn't like pineapple, she will take the share with the most orange.

Step 2 (Subdivision)

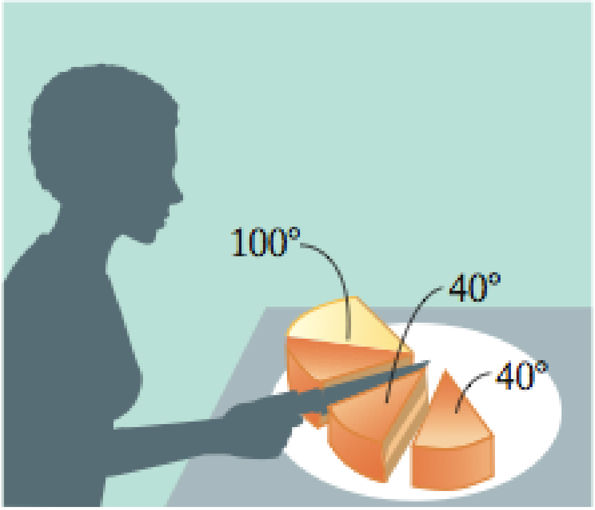
David divides his share into three subshares that in his opinion are of equal value (all the same size).



Example 3.7 Lone Chooser with 3 Players

Step 2 (Subdivision) continued

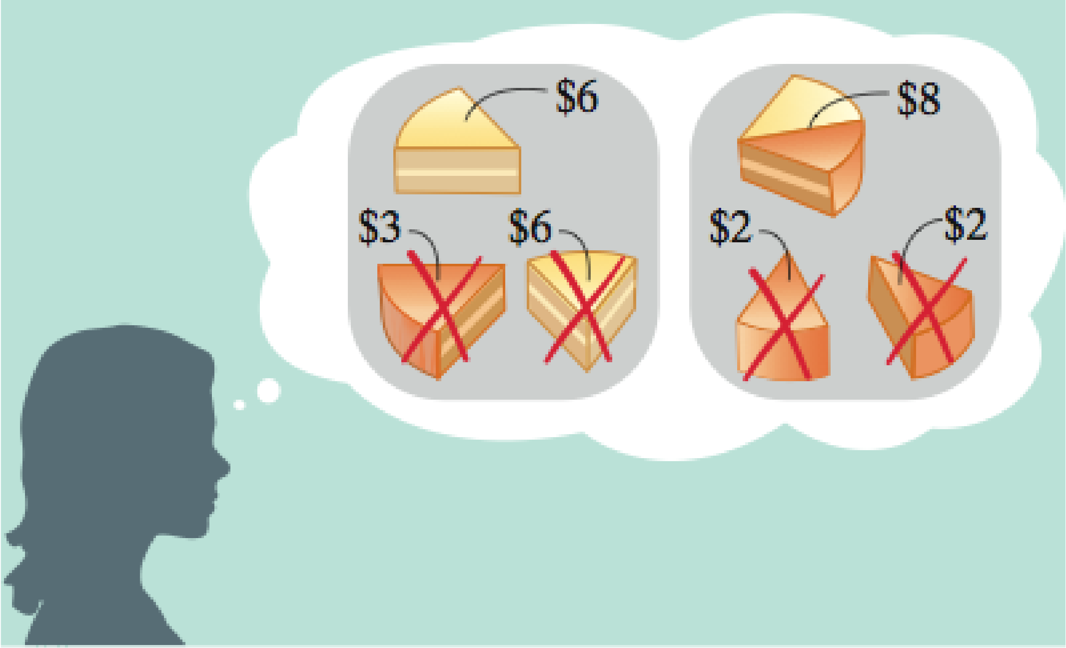
Dinah also divides her share into three smaller subshares that in her opinion are of equal value. (Remember that Dinah hates pineapple. Thus, she has made her cuts in such a way as to have one-third of the orange in each of the subshares.)



Example 3.7 Lone Chooser with 3 Players

Step 3 (Selection)

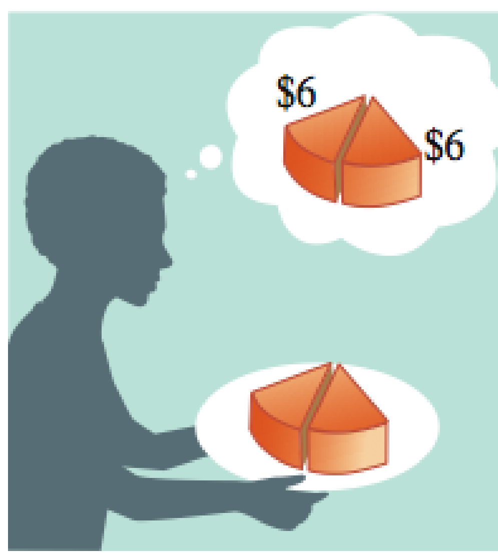
It's now Cher's turn to choose one sub-share from David's three and one subshare from Dinah's three. She will choose one of the two pineapple wedges from David's subshares and the big orange-pineapple wedge from Dinah's subshares.

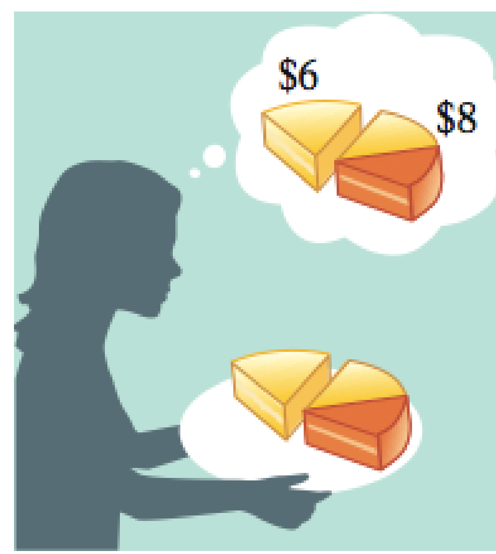


Example 3.7 Lone Chooser with 3 Players

Step 3 **(Selection)**

The final fair division of the cake is shown. David gets a final share worth $9, Dinah gets a final share worth $12, and Cher gets a final share worth $14. David is satisfied, Dinah is happy, and Cher is ecstatic.





The Lone-Chooser Method for *N* Players

In the general case of N players, the lone-chooser method involves one chooser C and $N - 1$ dividers D_1, D_2,…, D_{N-1}. As always, it is preferable to be a chooser than a divider, so the chooser is determined by a random draw. The method is based on an inductive strategy. If you can do it for three players, then you can do it for four players; if you can do it for four, then you can do it for five; and we can assume that we can use the lone-chooser method with $N - 1$ players.

The Lone-Chooser Method for *N* Players

Step 1 (Division)

D_1, D_2,…, D_{N-1} divide fairly the set *S* among themselves, as if *C* didn't exist. This is a fair division among *N* – 1 players, so each one gets a share he or she considers worth at least of 1/(*N* –1)th of *S*.

The Lone-Chooser Method for *N* Players

Step 2 (Subdivision)

Each divider subdivides his or her share into *N* sub-shares.

The Lone-Chooser Method for *N* Players

Step 3 (Selection)

The chooser *C* finally gets to play. *C* selects one sub-share from each divider – one subshare from D_1, one from D_2, and so on. At the end, *C* ends up with $N – 1$ subshares, which make up *C*'s final share, and each divider gets to keep the remaining $N – 1$ subshares in his or her subdivision.

The Lone-Chooser Method for *N* Players

When properly played, the lone-chooser method guarantees that everyone, dividers and chooser alike, ends up with a fair share